Solutions Exercise 1

1. $y = -2x^2 + 7x - 6$

$= -6 - [2x^2 - 7x]$

$= -6 - [2(x - \frac{7}{4})^2 - \frac{49}{8}]$

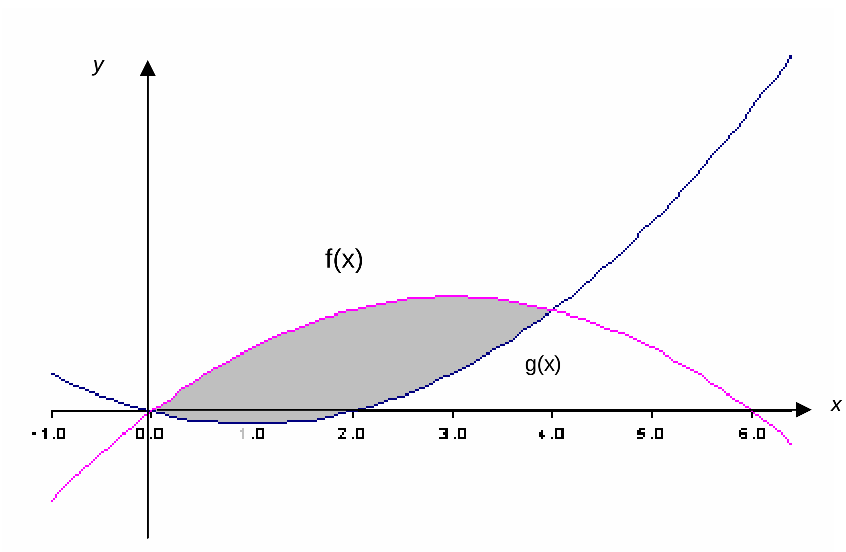
$= \frac{1}{8} - 2(x - \frac{7}{4})^2$ so max. T.P. $(\frac{7}{4}, \frac{1}{8})$

2. Let $\sin 75^o = \sin(45+30)^o = \sin(45)^0\cos(30)^0+\cos(45)^0\sin(30)^0$

$= \frac{1}{\sqrt{2}} x \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} x \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ as required.

Let $\sin 15^o = \sin(45-30)^o$ expand and substitute values as above = $\frac{\sqrt{3}-1}{2\sqrt{2}}$

3. $y = x^2 - 2x^2$ and $y = 6x - x^2$



Fist find the limits (the points of intersection of the two curves)

$x^2 – 2x^2 = 6x – x^2$
$2x^2 – 8x = 0$
$2x(x – 4) = 0$
$x = 0, x = 4$ *The limits are 0 and 4.*

The negative quadratic (sad face) will have a max TP so will be the upper curve

i.e. $f(x) = 6x – x^2$ *and* $g(x) = x^2 – 2x$

shaded area $= \int_0^4 f(x) - g(x)dx = \int_0^4 6x - x^2 - (x^2 - 2x)dx = \int_0^4 8x - 2x^2 dx$

$= \left[4x^2 - \frac{2x^3}{3}\right]_0^4$

$= (64 - \frac{128}{3}) = \frac{192 - 128}{3} = \frac{64}{3} = 21\frac{1}{3} units^2$

4. $P(r) = 2(r + \frac{16}{r})$ *radius r area* $16cm^2$

Perimeter P = r + r + arc length

$\frac{arc\_length}{2pr} = \frac{area\_sector}{pr^2}$ *substitute given values and cross multiply to obtain arc length*

arc length = $\frac{32}{r}$ *cm*

$P = r + r + \frac{32}{r} = 2(r + \frac{16}{r})$ *as required.*

A minimum value occurs at the minimum TP of this function.

A min. T.P. is a stationary point, and SPs occur when $\frac{dP}{dr} = 0$

$P = 2r + 32r^{-1}$

$\frac{dP}{dr} = 2 – 32r^{--2}$ *let this = 0 to find any SPs*

$2 - \frac{32}{r^2} = 0$ $r^2 = 16$ $r = 4$ *(since r is a distance it must be positive)*

To examine the nature of this SP examine $\frac{dP}{dr}$ *around r = 4*

r	$\rightarrow$	4	$\rightarrow$
$\frac{dP}{dr}\ (2 - \frac{32}{r^2})$	$-$	0	$+$
	$\setminus$	$\_\_$	$/$

minimum T.P. (4,16)

$P = 2(r + \frac{16}{r})$ *when* $r = 4$ $\qquad P = 16cm$

5. Show $x - 4$ is a factor of $2x^4 - 9x^3 + 5x^2 - 3x - 4$.

4	2	-9	5	-3	-4
		8	-4	4	4
	2	-1	1	1	0

since remainder = 0 then (x- 4) is a factor.

Factors $\qquad (x - 4)(2x^3 - x^2 + x + 1)$

6. A(1,2), B(5,6) and C(1,6)

$m_{AB} = \frac{6-2}{5-1} = 1$ $\qquad m_{AC}$ *undefined (same x co-ordinate so vertical line)*

$m_{BC} = 0$ *(horizontal line) So BC and AC are perpendicular.*

$AB^2 = (6 - 2)^2 + (5 - 1)^2 = 32 \qquad AC = \sqrt{32}$

$AC^2 = (6 - 2)^2 + (1 - 1)^2 = 16 \qquad AC = 4$

$BC^2 = (6 - 6)^2 + (1 - 5)^2 = 16 \qquad BC = 4$

Since $AB^2 = AC^2 + BC^2$ *then by the converse of Pythagoras ABC is a right-angled triangle.*

7. Show that $x + 3y + 1 = 0$ is a tangent to the circle $x^2 + y^2 + 10x + 4y + 19 = 0$.

Substitute $x = -3y - 1$ *into* $x^2 + y^2 + 10x + 4y + 19 = 0$.

$(-3y - 1)^2 + y^2 + 10(-3y - 1) + 4y + 19 = 0.$

$9y^2 + 6y + 1 + y^2 – 30y –10 + 4y + 19 = 0.$

$10y^2 – 20y + 10 = 0$

$10(y^2 – 2y + 1) = 0$ discriminant = 0 $\Rightarrow$ tangent

or

$10(y – 1)(y – 1) = 0$ equal roots $\Rightarrow$ tangent.

When y = 1 x = -3(1) – 1 = -4

Point of contact of line and circle is (-4,1).

8. $\tan(x – y) = \frac{\sin(x-y)}{\cos(x-y)} = \frac{\sin(x)\cos(y) - \cos(x)\sin(y)}{\cos(x)\cos(y) + \sin(x)\sin(y)}$ divide top and bottom by cos(y)

$= \frac{\sin(x) - \cos(x)\tan(y)}{\cos(x) + \sin(x)\tan(y)}$ divide top and bottom by cos(x)

$= \frac{\tan(x) - \tan(y)}{!+\tan(x)\tan(y)}$ as required.

9. $\tan x = \frac{1}{a-1}$ and $\tan y = \frac{1}{a+1}$ prove that $\tan(x\text{-}y) = \frac{2}{a^2}$

$\tan(x\text{-}y) = \frac{\frac{1}{a-1} - \frac{1}{a+1}}{1+\frac{1}{(a+1)(a-1)}}$ from above

$= \frac{\frac{a+1-(a-1)}{(a+1)(a-1)}}{\frac{(a+1)(a-1)+1}{(a+1)(a-1)}} = \frac{2}{a^2}$ as required

10. $f(x) = 2x^2 - x^4$, {x: $-\frac{1}{2} \le x \le \frac{1}{2}$}

The maximum and minimum values lie at either the end points of the interval or at any T.P. within the interval.

$f(-\frac{1}{2}) = \frac{1}{2} - \frac{1}{16} = \frac{7}{16}$ $\qquad f(\frac{1}{2}) = \frac{7}{16}$

Any T.P. is a S.P. and these occur when f'(x) = 0

$f'(x) = 4x - 4x^3$ *let this = 0 to find any S.P.'s*

$4x - 4x^3 = 0$

$4x(1 - x^2) = 0$

$x = 0$, $x = -1$, $x = 1$

Note that only one of these S.P.'s lie within the interval..

$f(0) = 0$

Within the given interval max. value is $\frac{7}{16}$ *and the min. value is 0.*

To sketch the graph of f(x) we require :

a) *roots*
b) *y – intercept*
c) *S.P.'s and their nature*
d) *examine f(x) as* $x \to +\infty$
e) *examine f(x) as* $x \to -\infty$

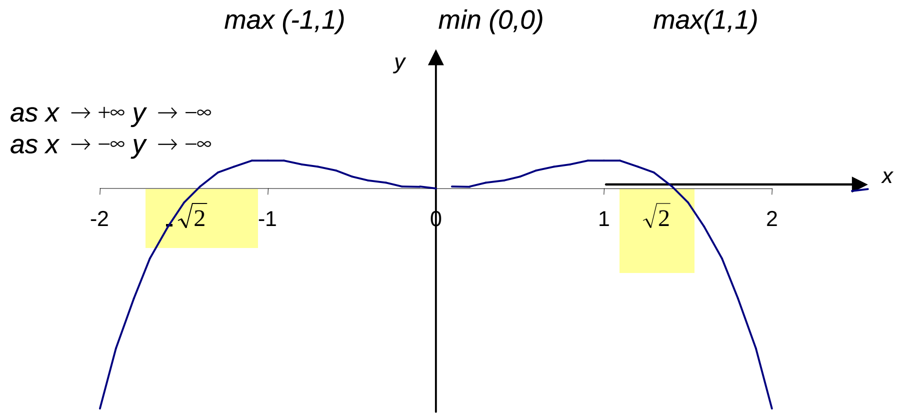
To find roots let f(x) = 0

$2x^2 - x^4 = 0$
$x^2(2 - x^2) = 0$
Roots at $x = 0$, $x = -\sqrt{2}$, $x = \sqrt{2}$

At y – intercept, x= 0 $\quad f(x) = 0$ *intercept (0,0)*

From above S.P.s occur at x = 0, x = -1, x = 1 To determine the nature examine f'(x) around these points.

x	→	-1	→	0	→	1	→
$f'(x)$ $4x(1 - x^2)$	+	0	-	0	+	0	-


max (-1,1)
min (0,0)
max(1,1)
y
as x → +∞ y → −∞
as x → −∞ y → −∞
x
-2
-√2
-1
0
1
√2
2